M-math 2nd year Final Exam
Subject : Probability Theory

Time : 3.00 hours Max.Marks 50.

1. Let $(X_t, \mathcal{F}_t)_{t\geq 0}$ be a super martingale. Let $\sigma \leq \tau$ be two bounded stopping times. Suppose $\sigma, \tau \in \{\frac{k}{2^n}; k = 0, 1, 2, \cdots, n = 1, 2, \cdots .\}$. Using the discrete time result show that $E[X_\tau|\mathcal{F}_\sigma] \leq X_\sigma, a.s.$ (10)

2. Let $(X_t)_{t\geq 0}$ be right continuous and $(\mathcal{F}_t)$ adapted. Then

(a) Show that (X_t) is progressively measurable i.e. for every $t \geq 0$ the map $(s, \omega) \to X_s(\omega) : [0, t] \times \Omega \to \mathbb{R}$ is $\mathcal{B}[0, t] \times \mathcal{F}_t$ measurable.

(b) Suppose $\tau : \Omega \to [0, \infty)$ is an $(\mathcal{F}_t)$ stopping time. Use part (a) to show that X_τ is $\mathcal{F}_\tau$ measurable.

(7+8)

3. Let $(X_t, \mathcal{F})_{t\geq 0}$ be a martingale or a non negative sub martingale with right continuous paths. Let $|X|^*_T := \sup_{t\leq T} |X_t|$. Show that

(a) For every $p \geq 1, \lambda > 0$

$$\lambda^p P\{|X|^*_T \geq \lambda\} \leq E|X_T|^p.$$

(b) For $p > 1$,

$$E|X|^*_T \leq (\frac{p}{p-1})^p E|X_T|^p.$$

Hint : by right continuity the supremum over $[0, T)$ of (X_t) is the supremum over the rationals in $[0, T)$. (6 + 6)

4. Let $(X_t, \mathcal{F})_{t\geq 0}$ be a sub martingale. Suppose that $\sup_{t\geq 0} EX_t^+ < \infty$. Show that there exists $X_\infty \in L^1$ such that $X_t \to X_\infty$ almost surely. Hint : Prove an upcrossing inequality and use that to show convergence. (13)